Cambridge Assessment
Admissions Testing

Sixth Term Examination Papers — **9470**

MATHEMATICS 2 — Morning

Thursday 8 June 2023 — Time: 3 hours

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully.

Do **NOT** open this question paper until you are told that you may do so.

Read and follow the additional instructions on the front of the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20.

You may answer as many questions as you choose. You are advised to concentrate on no more than six questions. Little credit will be given for fragmentary answers.

All your answers will be marked.

Crossed out work will NOT be marked.

Your final mark will be based on the six questions for which you gain the highest marks.

There is NO Mathematical Formulae Booklet.

Calculators are NOT permitted.

Bilingual dictionaries are NOT permitted.

Wait to be told you may begin before turning this page.

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Section A: Pure Mathematics

1 **(i)** Show that making the substitution $x = \dfrac{1}{t}$ in the integral

$$\int_a^b \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x\,,$$

where $b > a > 0$, gives the integral

$$\int_{a^{-1}}^{b^{-1}} \frac{-t}{(1+t^2)^{\frac{3}{2}}}\,\mathrm{d}t\,.$$

(ii) Evaluate:

(a)
$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x\,;$$

(b)
$$\int_{-2}^{2} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x\,.$$

(iii) **(a)** Show that

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \int_{\frac{1}{2}}^{2} \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = \tfrac{1}{2}\int_{\frac{1}{2}}^{2} \frac{1}{1+x^2}\,\mathrm{d}x\,,$$

and hence evaluate

$$\int_{\frac{1}{2}}^{2} \frac{1}{(1+x^2)^2}\,\mathrm{d}x\,.$$

(b) Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{1-x}{x(1+x^2)^{\frac{1}{2}}}\,\mathrm{d}x\,.$$

2 **(i)** The real numbers x, y and z satisfy the equations

$$\begin{aligned} y &= \frac{2x}{1-x^2}\,, \\ z &= \frac{2y}{1-y^2}\,, \\ x &= \frac{2z}{1-z^2}\,. \end{aligned}$$

Let $x = \tan\alpha$. Deduce that $y = \tan 2\alpha$ and show that $\tan\alpha = \tan 8\alpha$.

Find all solutions of the equations, giving each value of x, y and z in the form $\tan\theta$ where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(ii) Determine the number of real solutions of the simultaneous equations

$$\begin{aligned} y &= \frac{3x-x^3}{1-3x^2}\,, \\ z &= \frac{3y-y^3}{1-3y^2}\,, \\ x &= \frac{3z-z^3}{1-3z^2}\,. \end{aligned}$$

(iii) Consider the simultaneous equations

$$\begin{aligned} y &= 2x^2 - 1\,, \\ z &= 2y^2 - 1\,, \\ x &= 2z^2 - 1\,. \end{aligned}$$

(a) Determine the number of real solutions of these simultaneous equations with $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$.

(b) By finding the degree of a single polynomial equation which is satisfied by x, show that all solutions of these simultaneous equations have $|x| \leqslant 1$, $|y| \leqslant 1$, $|z| \leqslant 1$.

3 Let $\mathrm{p}(x)$ be a polynomial of degree n with $\mathrm{p}(x) > 0$ for all x and let

$$\mathrm{q}(x) = \sum_{k=0}^{n} \mathrm{p}^{(k)}(x)\,,$$

where $\mathrm{p}^{(k)}(x) \equiv \dfrac{\mathrm{d}^k \mathrm{p}(x)}{\mathrm{d}x^k}$ for $k \geqslant 1$ and $\mathrm{p}^{(0)}(x) \equiv \mathrm{p}(x)$.

(i) **(a)** Explain why n must be even and show that $\mathrm{q}(x)$ takes positive values for some values of x.

(b) Show that $\mathrm{q}'(x) = \mathrm{q}(x) - \mathrm{p}(x)$.

(ii) In this part you will be asked to show the same result in three different ways.

(a) Show that the curves $y = \mathrm{p}(x)$ and $y = \mathrm{q}(x)$ meet at every stationary point of $y = \mathrm{q}(x)$.

Hence show that $\mathrm{q}(x) > 0$ for all x.

(b) Show that $\mathrm{e}^{-x}\mathrm{q}(x)$ is a decreasing function.

Hence show that $\mathrm{q}(x) > 0$ for all x.

(c) Show that

$$\int_0^\infty \mathrm{p}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{p}(x) + \int_0^\infty \mathrm{p}^{(1)}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t\,.$$

Show further that

$$\int_0^\infty \mathrm{p}(x+t)\mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{q}(x)\,.$$

Hence show that $\mathrm{q}(x) > 0$ for all x.

4 **(i)** Show that, if $\left(x - \sqrt{2}\right)^2 = 3$, then $x^4 - 10x^2 + 1 = 0$.

Deduce that, if $\mathrm{f}(x) = x^4 - 10x^2 + 1$, then $\mathrm{f}\left(\sqrt{2} + \sqrt{3}\right) = 0$.

(ii) Find a polynomial g of degree 8 with integer coefficients such that $\mathrm{g}\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right) = 0$. Write your answer in a form without brackets.

(iii) Let a, b and c be the three roots of $t^3 - 3t + 1 = 0$.

Find a polynomial h of degree 6 with integer coefficients such that $\mathrm{h}\left(a + \sqrt{2}\right) = 0$, $\mathrm{h}\left(b + \sqrt{2}\right) = 0$ and $\mathrm{h}\left(c + \sqrt{2}\right) = 0$. Write your answer in a form without brackets.

(iv) Find a polynomial k with integer coefficients such that $\mathrm{k}\left(\sqrt[3]{2} + \sqrt[3]{3}\right) = 0$. Write your answer in a form without brackets.

5 **(i)** The sequence x_n for $n = 0, 1, 2, \ldots$ is defined by $x_0 = 1$ and by

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}$$

for $n \geqslant 0$.

(a) Explain briefly why $x_n \geqslant 1$ for all n.

(b) Show that $x_{n+1}^2 - 2$ and $x_n^2 - 2$ have opposite sign, and that

$$\left|x_{n+1}^2 - 2\right| \leqslant \tfrac{1}{4}\left|x_n^2 - 2\right| .$$

(c) Show that

$$2 - 10^{-6} \leqslant x_{10}^2 \leqslant 2 .$$

(ii) The sequence y_n for $n = 0, 1, 2, \ldots$ is defined by $y_0 = 1$ and by

$$y_{n+1} = \frac{y_n^2 + 2}{2y_n}$$

for $n \geqslant 0$.

(a) Show that, for $n \geqslant 0$,

$$y_{n+1} - \sqrt{2} = \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}$$

and deduce that $y_n \geqslant 1$ for $n \geqslant 0$.

(b) Show that

$$y_n - \sqrt{2} \leqslant 2\left(\frac{\sqrt{2} - 1}{2}\right)^{2^n}$$

for $n \geqslant 1$.

(c) Using the fact that

$$\sqrt{2} - 1 < \tfrac{1}{2} ,$$

or otherwise, show that

$$\sqrt{2} \leqslant y_{10} \leqslant \sqrt{2} + 10^{-600} .$$

6 The sequence F_n, for $n = 0, 1, 2, \ldots$, is defined by $F_0 = 0$, $F_1 = 1$ and by $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$.

Prove by induction that, for all positive integers n,

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \mathbf{Q}^n,$$

where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) By considering the matrix $\mathbf{Q}^n$, show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all positive integers n.

(ii) By considering the matrix $\mathbf{Q}^{\mathbf{m+n}}$, show that $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ for all positive integers m and n.

(iii) Show that $\mathbf{Q}^2 = \mathbf{I} + \mathbf{Q}$.

In the following parts, you may use without proof the Binomial Theorem for matrices:

$$(\mathbf{I} + \mathbf{A})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k.$$

(a) Show that, for all positive integers n,

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k\,.$$

(b) Show that, for all positive integers n,

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k$$

and also that

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}\,.$$

(c) Show that, for all positive integers n,

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{n+k} = 0\,.$$

7 **(i)** The complex numbers z and w have real and imaginary parts given by $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$. Prove that $|zw| = |z||w|$.

(ii) By considering the complex numbers $2 + \mathrm{i}$ and $10 + 11\mathrm{i}$, find positive integers h and k such that $h^2 + k^2 = 5 \times 221$.

(iii) Find positive integers m and n such that $m^2 + n^2 = 8045$.

(iv) You are given that $102^2 + 201^2 = 50805$.

Find positive integers p and q such that $p^2 + q^2 = 36 \times 50805$.

(v) Find three distinct pairs of positive integers r and s such that $r^2 + s^2 = 25 \times 1002082$ and $r < s$.

(vi) You are given that $109 \times 9193 = 1002037$.

Find positive integers t and u such that $t^2 + u^2 = 9193$.

8 A tetrahedron is called isosceles if each pair of edges which do not share a vertex have equal length.

(i) Prove that a tetrahedron is isosceles if and only if all four faces have the same perimeter.

Let $OABC$ be an isosceles tetrahedron and let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.

(ii) By considering the lengths of OA and BC, show that

$$2\mathbf{b}.\mathbf{c} = |\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2.$$

Show that

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = |\mathbf{a}|^2.$$

(iii) Let G be the *centroid* of the tetrahedron, defined by $\overrightarrow{OG} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

Show that G is equidistant from all four vertices of the tetrahedron.

(iv) By considering the length of the vector $\mathbf{a} - \mathbf{b} - \mathbf{c}$, or otherwise, show that, in an isosceles tetrahedron, none of the angles between pairs of edges which share a vertex can be obtuse. Can any of them be right angles?

Section B: Mechanics

9 A truck of mass M is connected by a light, rigid tow-bar, which is parallel to the ground, to a trailer of mass kM. A constant driving force D which is parallel to the ground acts on the truck, and the only resistance to motion is a frictional force acting on the trailer, with coefficient of friction μ.

- When the truck pulls the trailer up a slope which makes an angle α to the horizontal, the acceleration is a_1 and there is a tension T_1 in the tow-bar.
- When the truck pulls the trailer on horizontal ground, the acceleration is a_2 and there is a tension T_2 in the tow-bar.
- When the truck pulls the trailer down a slope which makes an angle α to the horizontal, the acceleration is a_3 and there is a tension T_3 in the tow-bar.

All accelerations are taken to be positive when in the direction of motion of the truck.

(i) Show that $T_1 = T_3$ and that $M(a_1 + a_3 - 2a_2) = 2(T_2 - T_1)$.

(ii) It is given that $\mu < 1$.

(a) Show that

$$a_2 < \tfrac{1}{2}(a_1 + a_3) < a_3\,.$$

(b) Show further that

$$a_1 < a_2\,.$$

10 In this question, the x- and y-axes are horizontal and the z-axis is vertically upwards.

(i) A particle P_α is projected from the origin with speed u at an acute angle α above the positive x-axis.

The curve E is given by $z = A - Bx^2$ and $y = 0$. If E and the trajectory of P_α touch exactly once, show that

$$u^2 - 2gA = u^2\big(1 - 4AB\big)\cos^2\alpha\,.$$

E and the trajectory of P_α touch exactly once for all α with $0 < \alpha < \frac{1}{2}\pi$. Write down the values of A and B in terms of u and g.

An explosion takes place at the origin and results in a large number of particles being simultaneously projected with speed u in different directions. You may assume that all the particles move freely under gravity for $t \geqslant 0$.

(ii) Describe the set of points which can be hit by particles from the explosion, explaining your answer.

(iii) Show that, at a time t after the explosion, the particles lie on a sphere whose centre and radius you should find.

(iv) Another particle Q is projected horizontally from the point $(0, 0, A)$ with speed u in the positive x direction.

Show that, at all times, Q lies on the curve E.

(v) Show that for particles Q and P_α to collide, Q must be projected a time $\dfrac{u(1-\cos\alpha)}{g\sin\alpha}$ after the explosion.

Section C: Probability and Statistics

11 **(i)** X_1 and X_2 are both random variables which take values $x_1, x_2, \ldots, x_n$, with probabilities $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively.

The value of random variable Y is defined to be that of X_1 with probability p and that of X_2 with probability $q = 1 - p$.

If X_1 has mean μ_1 and variance σ_1^2, and X_2 has mean μ_2 and variance σ_2^2, find the mean of Y and show that the variance of Y is $p\sigma_1^2 + q\sigma_2^2 + pq(\mu_1 - \mu_2)^2$.

(ii) To find the value of random variable B, a fair coin is tossed and a fair six-sided die is rolled. If the coin shows heads, then $B = 1$ if the die shows a six and $B = 0$ otherwise; if the coin shows tails, then $B = 1$ if the die does **not** show a six and $B = 0$ if it does. The value of Z_1 is the sum of n independent values of B, where n is large.

Show that Z_1 is a Binomial random variable with probability of success $\frac{1}{2}$.

Using a Normal approximation, show that the probability that Z_1 is within 10% of its mean tends to 1 as $n \longrightarrow \infty$.

(iii) To find the value of random variable Z_2, a fair coin is tossed and n fair six-sided dice are rolled, where n is large. If the coin shows heads, then the value of Z_2 is the number of dice showing a six; if the coin shows tails, then the value of Z_2 is the number of dice **not** showing a six.

Use part **(i)** to write down the mean and variance of Z_2.

Explain why a Normal distribution with this mean and variance will not be a good approximation to the distribution of Z_2.

Show that the probability that Z_2 is within 10% of its mean tends to 0 as $n \longrightarrow \infty$.

12 Each of the independent random variables $X_1, X_2, \ldots, X_n$ has the probability density function $\mathrm{f}(x) = \frac{1}{2}\sin x$ for $0 \leqslant x \leqslant \pi$ (and zero otherwise). Let Y be the random variable whose value is the maximum of the values of $X_1, X_2, \ldots, X_n$.

(i) Explain why $\mathrm{P}(Y \leqslant t) = \big[\mathrm{P}(X_1 \leqslant t)\big]^n$ and hence, or otherwise, find the probability density function of Y.

Let $m(n)$ be the median of Y and $\mu(n)$ be the mean of Y.

(ii) Find an expression for $m(n)$ in terms of n. How does $m(n)$ change as n increases?

(iii) Show that

$$\mu(n) = \pi - \frac{1}{2^n}\int_0^\pi (1 - \cos x)^n \, \mathrm{d}x \,.$$

(a) Show that $\mu(n)$ increases with n.

(b) Show that $\mu(2) < m(2)$.

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